A 1

a) $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

$\sin(y)\cos(x) + \sin(x)\cos(y)$

$= \frac{1}{4i}\left((e^{iy}-e^{-iy})(e^{ix}+e^{-ix}) + (e^{ix}-e^{-ix})(e^{iy}+e^{-iy})\right)$

$= \frac{1}{4i}\left(e^{i(x+y)} + e^{i(y-x)} - e^{i(x-y)} - e^{-i(x+y)} + e^{i(x+y)} - e^{i(y-x)} + e^{i(x-y)} - e^{-i(x+y)}\right)$

$= \frac{1}{4i}\left(2(e^{i(x+y)} - e^{-i(x+y)})\right) = 2\sin(x+y)$

b) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x)\cos(y) - \sin(x)\sin(y) =$

$\frac{1}{4}(e^{ix}+e^{-ix})(e^{iy}+e^{-iy}) - \frac{1}{4i^2}(e^{ix}-e^{-ix})(e^{iy}-e^{-iy})$

$\frac{1}{4}(e^{i(x+y)} + e^{i(x-y)} + e^{i(y-x)} + e^{-i(x+y)}) + \frac{1}{4}(e^{i(x+y)} - e^{i(x-y)} - e^{i(y-x)} + e^{-i(x+y)})$

$= \frac{1}{4}(2e^{i(x+y)} + 2e^{-i(x+y)}) = \cos(x+y)$

c) $1+\cot^2 = \frac{1}{\sin^2}$

$1+\cot^2 = 1 + \left(\frac{\cos}{\sin}\right)^2 = \frac{\sin^2}{\sin^2} + \frac{\cos^2}{\sin^2} = \frac{\cos^2+\sin^2}{\sin^2}$

$= \frac{1}{\sin^2}$

d) $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} =$

$\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin y \sin x}{\cos y \cos x}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{\frac{\cos x\cos y - \sin x \sin y}{\cos y \cos x}}$

$= \frac{\sin x\cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} = \frac{\sin(x+y)}{\cos(x+y)} = \tan(x+y)$

A2

a) $$\lim_{x\to 0} \frac{x^3}{\sin(x)-x} = \lim_{x\to 0} \frac{x^3}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - x}$$

$$= \lim_{x\to 0} \frac{x^3}{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}} = \lim_{x\to 0} \frac{1}{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2(n-1)}}$$

Der Grenzwert der Reihe existiert, also ist die Funktion stetig und es gilt

$$= \frac{1}{\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2(n-1)}} = \frac{1}{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2(n-1)}} = \frac{1}{-\frac{1}{3!}} = -6$$

b) $$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{\exp(x^4) - 1} = \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} - 1 + \frac{x^2}{2}}{\sum_{n=0}^{\infty} \frac{1}{n!} x^{4n} - 1}$$

$$= \lim_{x\to 0} \frac{\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \cdot \frac{1}{x^4}}{\sum_{n=1}^{\infty} \frac{1}{n!} x^{4n} \cdot \frac{1}{x^4}} = \lim_{x\to 0} \frac{\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2(n-2)}}{\sum_{n=1}^{\infty} \frac{1}{n!} x^{4(n-1)}} =$$

$$= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+4)!} x^{2n}}{\sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{4n}} \xrightarrow[x\to 0]{} \frac{1/(4!)}{1/1!} = \frac{1}{24}$$

A3

a) f sei gerade:

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \overset{h' = -h}{=} \lim_{-h' \to 0} \frac{f(-x-h') - f(-x)}{-h'}$$

$$\underset{\text{gerade}}{=} \lim_{-h' \to 0} \frac{f(x+h') - f(x)}{-h'} \underset{\lim_{h\downarrow 0} = \lim_{h \uparrow 0}}{=} \lim_{h' \to 0} \frac{f(x+h') - f(x)}{-h'} = -f'(x)$$

b) f sei ungerade

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \overset{h' = -h}{=} \lim_{-h' \to 0} \frac{f(-x-h') - f(-x)}{-h'}$$

$$\underset{\text{ungerade}}{=} \lim_{-h' \to 0} \frac{-f(x+h') + f(x)}{-h'} \underset{\lim_{h\downarrow 0} = \lim_{h \uparrow 0}}{=} \lim_{h' \to 0} \frac{f(x+h') - f(x)}{h'} = f'(x)$$

A4 finde die Ableitung zu:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{falls } x \neq 0 \\ 0 & \text{falls } x = 0 \end{cases}$$

- Auf $[-1, 1] \setminus \{0\}$ gilt $f(x) = x^2 \sin(\frac{1}{x})$

also, für $x \neq 0$, $f'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2})$
$= 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$

- Bei $x = 0$ definiere die differenzenquotienten-funktion $d: [-1,1] \setminus \{0\} \to \mathbb{R}$

$$d_0(h) = \frac{f(h) - f(0)}{h} = \frac{h^2 \sin(\frac{1}{h})}{h} = h \sin(\frac{1}{h})$$

Wegen $|\sin \frac{1}{h}| \leq 1$ für alle $h \neq 0$ gilt, für $\varepsilon > 0$,

$|d_0(\delta) - 0| = |\delta \sin(\frac{1}{\delta})| \leq \delta$

Also gilt $d_0(h) \xrightarrow[h \to 0]{} 0$, also exeistiert $f'(0)$ und $f'(0) = 0$

Zusammen:

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & \text{falls } x \neq 0 \\ 0 & \text{falls } x = 0 \end{cases}$$

Ist die Ableitung stetig? Nein! denn:

Sei $\varepsilon = \frac{1}{2}$, $\delta > 0$. Wähle $n \in \mathbb{N}$ so, dass $2\pi n > 1/\delta$ gilt. Also: $f'(\frac{1}{2\pi n}) = \frac{\sin(2\pi n)}{2\pi n} - \cos(2\pi n) = -1$

$|f'(\frac{1}{2\pi n})| = 1 > \varepsilon$ obwohl $|\frac{1}{2\pi n}| < \delta$ für alle $\delta > 0$

A5

Untersuche die folgenden Funktionen auf diffbarkeit bei $x=0$

a) $f: \mathbb{R}_0^+ \to \mathbb{R}, \ x \mapsto \sqrt[3]{x}$

$$\lim_{h \downarrow 0} \frac{f(h)-f(0)}{h} = \lim_{h \downarrow 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \downarrow 0} \sqrt[3]{\frac{1}{h^2}}$$

$= \infty$, da $\frac{1}{h^2} \xrightarrow[h \to 0]{} \infty$

$\hookrightarrow f'(0)$ existiert nicht.

b) $f: \mathbb{R} \to \mathbb{R}, \ x \to \sqrt{|x|}$

$$\lim_{h \downarrow 0} \frac{f(h)-f(0)}{h} = \lim_{h \downarrow 0} \frac{\sqrt{h}}{h} = \lim_{h \downarrow 0} \frac{1}{\sqrt{h}} = \infty$$

da $\frac{1}{\sqrt{h}} \xrightarrow[h \to 0]{} \infty$

$\hookrightarrow f'(0)$ existiert nicht (Egal, was $\lim_{h \uparrow 0} \frac{f(h)-f(0)}{h}$ ergibt)

c) $f: \mathbb{R} \to \mathbb{R}, \ x \mapsto x \cdot |x|$

Nach 6.1.13 und 6.1.14:

$f|_{[0,\infty)} = x^2 \quad \Rightarrow f'_+(0) = 0$

$f|_{(-\infty,0]} = -x^2 \quad \Rightarrow f'_-(0) = 0$

wegen $f'_-(0) = f'_+(0) = 0$ existiert $f'(0)$ mit

$f'(0) = 0$